SHORT COURSE
IN
SPHERICAL TRIGONOMETRY

SPERRY

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SHORT COURSE
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SPHERICAL TRIGONOMETRY

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BY

PAULINE SPERRY, PH.D.
ASSISTANT PROFESSOR OF MATHEMATICS
UNIVERSITY OF CALIFORNIA

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PREFACE

THERE seems to be a real need for a short yet rigorous text in spherical trigonometry which shall contain little more theory than is needed for the solution of spherical triangles, with a few applications to add interest to the subject. This book is the result of an effort to meet that need. As there are often but few class hours to be devoted to the subject, the aim of the writer has been to present matters in such a way as to enable the student to master the elements with little help from outside. The average student finds it hard to do this when he must pick out the essentials for himself.

A knowledge of spherical geometry is not presupposed, but a brief presentation of the principal concepts and theorems is given in Chapter I. Most of the theoretical examples bear directly on the development of the theory in the text. The data for the numerical solution of triangles are such as to require a minimum of labor in computation while offering a wider variety of applications of the theory than is usual in problem sets. Especial attention is called to the summarization of the tests for the number of solutions in the ambiguous cases in four short and complete tables, material usually covering several pages and often incomplete. The student is encouraged to check all solutions, and methods of checking are given in every case.

The time required for the discussion and solution of the general spherical triangle may be reduced by half by considering only Cases 1, 2, and 3 in Chapter III, explaining how the other cases may be solved by means of the polar triangle.

The terminology and symbols used follow the recommendations in the *Report of the National Committee on Mathematical Requirements* under the auspices of the Mathematical Association of America (1923).

A brief history of the subject based on Cajori's *History of Mathematics*, Braunmühl's *Vorlesungen über die Geschichte der Trigonometrie*, and Tröpfke's *Geschichte der Elementarmathematik* (1921) is included. The theory underlying the problems in astronomy is based on W. W. Campbell's *Practical Astronomy* and F. R. Moulton's *Introduction to Astronomy*. Some of the older texts on spherical trigonometry have also been consulted, in particular those of Chauvenet, Murray, and Todhunter.

The writer wishes to acknowledge her indebtedness to Professor E. J. Wilczynski and to the late Professor Eleanor P. Cushing for their careful reading of the text and for their valuable suggestions; to Professor H. E. Slaught for his interest in its publication; to Professor Florian Cajori for reading that part of the text relating to history; to Professor G. D. Shane for reading the parts relating to astronomy; and to Mr. Edward Roessler for enlarging the figures for printing.

PAULINE SPERRY

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SHORT COURSE IN SPHERICAL TRIGONOMETRY

CHAPTER I

CONCERNING SOLID AND SPHERICAL GEOMETRY

I. PLANES AND LINES IN SPACE

The theorems of this section are given (without proof) as a basis for a more detailed treatment of the geometry of the sphere in the next section.

1. Lines perpendicular to a plane. Definition. A line is **perpendicular** to a plane if it is perpendicular to every line lying in the plane and passing through its foot.

Theorem. *If a line is perpendicular to two lines at their point of intersection, it is perpendicular to the plane of the two lines.*

2. Parallel planes. Definition. Two planes are **parallel** if they never meet.

Theorem 1. *Two non-parallel planes meet in a straight line.*

Theorem 2. *If two parallel planes are cut by a third plane, the intersections are parallel lines.*

Theorem 3. *Two planes perpendicular to the same straight line are parallel.*

Theorem 4. *Two angles not in the same plane and having their sides respectively parallel in the same sense are equal and their planes are parallel.*

3. Perpendicular planes. Definitions. The space angle which one of two intersecting planes makes with the other is called a **dihedral angle.** The planes are the **faces** and the line of intersection

is the **edge** of the angle. If a plane is passed perpendicular to the edge at any point, its intersections with the faces form a plane angle which is defined as the **measure** of the dihedral angle.

Ex. Show that the measure of the dihedral angle is independent of the position of the cutting plane.

Theorem 1. *If a plane is perpendicular to a line, it is also perpendicular to every plane which passes through the line.*

Theorem 2. *If two planes are perpendicular, a line drawn in one of them perpendicular to their intersection is also perpendicular to the other plane.*

Theorem 3. *If two planes are perpendicular, a line perpendicular to the first from a point in the second lies wholly in the second.*

Theorem 4. *If two intersecting planes are each perpendicular to a third, so also is their intersection.*

II. CERTAIN PROPERTIES OF THE SPHERE

A clear understanding of some of the properties of a sphere is essential to the study of spherical trigonometry. Methods of proving the theorems of this section are suggested where proofs are not given in detail. An exception is made in the case of theorems which can easily be proved by the student.

4. Definitions. A **sphere** is a solid bounded by a surface all points of which are equally distant from a point within called the **center.** The line joining the center to any point on the surface is a **radius** of the sphere. The line segment drawn through the center and having its end points in the surface is a **diameter** of the sphere.

5. Circles on a sphere. Theorem 1. *Every section of a spherical surface made by a plane is a circle.*

Suppose that the plane does not pass through the center of the sphere. Let A and B be any two points on the boundary of the section and O be the center of the sphere. Drop a perpendicular OM to the cutting plane. Draw the radii OA and OB and the lines MA and MB. The right triangles OMA and OMB are equal, so MA equals MB for every A and B. Therefore the section is a

circle having M as its center. Such a section is called a **small circle** of the sphere.

If the plane passes through the center of the sphere, the theorem is obvious and the section is a **great circle** whose center and radius are those of the sphere.

Theorem 2. *All great circles are equal.*

Theorem 3. *Two great circles bisect each other.*

6. Length of arc of a circle. Since three non-collinear points determine a plane, a unique great circle is determined by the center of the sphere and any two points, A and B, on the surface which are not the extremities of a diameter. Obviously two arcs, AB and BA, are obtained, one greater and one less than a semicircumference. When we speak of the arc AB, we shall always mean the smaller arc. It can be proved that the shortest path from one point to another on a spherical surface is the arc of a great circle. So the arc of a great circle joining two points is the natural measure of spherical distance. Distance may therefore be given in circular or angular measure and is numerically equal to the measure of the angle at the center of the sphere which is subtended by the arc.

EXERCISES

1. How many small circles are there through any two points, A and B, on the surface of a sphere? Are the arcs AB the same in length for all these circles?

2. Show that if l be the length of the arc AB measured in the same linear units as the radius r of the sphere, and a be the distance in radians, then $l = ar$.

7. Poles of a circle. Definitions. An **axis** of any circle of a sphere is the diameter which is perpendicular to the plane of the circle. The extremities of the axis are the **poles.**

Theorem 1. *A pole of a circle is equally distant from all points on the circumference.*

This constant distance on the surface of a sphere is called the **polar distance.** The polar distance of a great circle is 90° and is called a **quadrant.**

Theorem 2. *If a point on the surface of a sphere is at a quadrant's distance from each of two other points, not the extremities of a diameter, then it is the pole of the great circle passing through the two points.*

To prove this, use the theorem in article **1.**

Theorem 3. *If two great circles are perpendicular to each other, each passes through the poles of the other.* (See art. **3,** th. 3.)

8. Spherical angles. Definitions. The intersecting arcs of two great circles form a **spherical angle.** The **measure** of this angle is the measure of the dihedral angle formed by the planes of the circles. A **right** spherical angle is one whose measure is 90°. Two arcs are **perpendicular** if they form a right spherical angle.

Theorem. *The measure of a spherical angle is equal to the arc of a great circle described from its vertex as a pole and included between its sides produced if necessary.*

EXERCISES

1. Prove that through a given point an arc of a great circle can be passed perpendicular to a given great circle and that the arc is unique provided the point is not the pole of the circle. (See art. **3,** th. 1.)

2. Prove that a spherical angle has the same measure as the angle formed by the tangents to the two arcs at the vertex. (See art. **1,** th. 4.)

3. Use the last example to prove that the sum of all the spherical angles about a point is 360°.

4. Show that each side of a right spherical angle passes through the poles of the other.

9. The spherical triangle. Definitions. If any number of points $A, B, C, \cdots, N$ on the surface of a sphere are joined in succession by the arcs of great circles $AB, BC, CD, \cdots, NA$, the figure so formed is a **spherical polygon.** The enclosing arcs are the **sides** and the spherical angles at $A, B, C, \cdots, N$ are the **angles** of the polygon. As in plane geometry, a **regular** polygon has equal sides and equal angles. A **spherical triangle** is a spherical polygon having three sides. Its angles are denoted by A, B, C and the sides opposite by a, b, c respectively. **Spherical trigonometry** is concerned with relations between the sides and angles of a spherical triangle.

In order to conform to common usage and so avoid ambiguity, let us agree that the sides of a triangle be each less than a semicircle. So while in the adjoining figure, one might consider $AEBC$ as a spherical triangle * having its vertices at A, B, and C and as its sides BC, CA, and AEB, such figures will be excluded from the class of triangles by this agreement.

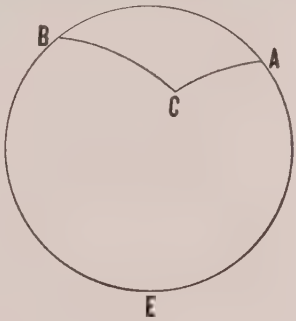

FIG. 1

Theorem 1. *Each side of a spherical triangle is less than the sum of the other two sides.* (See art. **6.**)

Theorem 2. *The sum of the sides of a spherical triangle is less than 360°.*

For if two great circles intersecting at A and A', the extremities of a diameter, are cut by a third circle in B and C, then $BC < A'B + A'C = (180° - AB) + (180° - AC)$. Therefore $a < 360° - c - b$, or $a + b + c < 360°$.

Definitions. Obviously a spherical triangle may have one, two, or all of its sides greater than, or equal to, or less than a quadrant. A triangle is **quadrantal, biquadrantal,** or **triquadrantal,** according as one, two, or three of its sides are quadrants.

Spherical triangles may be **equilateral, equiangular, isosceles, right,** or **oblique.** These notions are analogous to the corresponding ones in plane geometry, excepting that a right spherical triangle may have more than one right angle, and an oblique triangle more than one oblique angle. A triangle is **rectangular, birectangular,** or **trirectangular** according as it has one, two, or three right angles.

10. Relations of sides and angles in spherical triangles.

Theorem 1. *The base angles of an isosceles spherical triangle are equal.*

Let b and c be the equal sides and O the center of the sphere. Draw OA, OB, and OC. Also draw BS and CS tangent to a at B

* Gauss was the first to suggest the use of such triangles in his *Theoria Motus Corporum Coelestium*, 1809. For discussion of these triangles see Chauvenet's *Spherical Trigonometry.*

and C respectively. These tangents meet at T, a point of OA extended, because the planes of the arcs b and c intersect in OA. Draw TS. Then triangle OCT equals triangle OBT. So CT equals BT, and triangle CST equals triangle BST. Since angle SCT equals angle SBT, in the spherical triangle angle B equals angle C. (See art. **8,** ex. 2.)

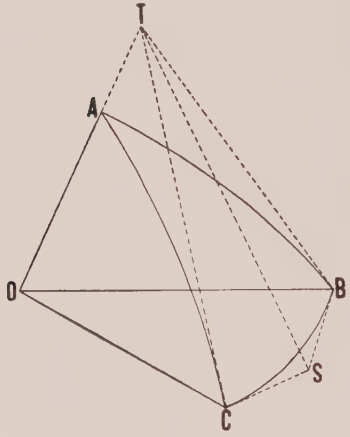

FIG. 2

In Figure 2, b and c were taken less than quadrants. Let the student draw the figure and make the necessary modifications in the proof if the equal sides are greater than quadrants. If b and c are quadrants, the base angles are right angles (art. **1,** th.; art. **3,** th. 1) and the third side is the measure of the angle opposite (art. **7,** th. 2; art. **8,** th.).

Ex. Prove that *an equilateral spherical triangle is also equiangular.*

Theorem 2. *If two angles of a spherical triangle are unequal, that side is greater which is opposite the greater angle.*

The proof of this theorem is similar to the corresponding one in plane geometry.

11. Polar triangles. Definition. Let ABC be a spherical triangle. Then each side of the triangle has two poles. Let A' be that pole of a which is on the same side of it as A. Let B' and C' be similarly defined for b and c respectively. Then AA', BB', and CC' are each less than a quadrant. Join $A'B'$, $B'C'$, and $C'A'$ by the arcs of great circles. $A'B'C'$ is defined as the **polar triangle** of ABC. Denote its sides by a', b', and c'.

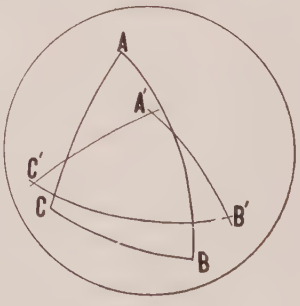

FIG. 3

Theorem 1. *If $A'B'C'$ is the polar triangle of ABC, then reciprocally ABC is the polar triangle of $A'B'C'$.*

For since B' and C' are the poles of b and c respectively, they are each at a quadrant's distance from A, the intersection of b and c. Therefore A is a pole of $B'C'$ (art. **7**, th. 2). Furthermore A is on the same side of $B'C'$ as A', for AA' is less than a quadrant. Similarly B and C are the poles of $A'C'$ and $A'B'$ and are on the same sides of them as B' and C' respectively. Hence ABC is the polar triangle of $A'B'C'$.

Ex. What is the polar triangle of a triquadrantal triangle? of a trirectangular triangle?

Theorem 2. *In polar triangles each side of one is the supplement of the angle at its pole in the other.*

For let a' and c' (produced if necessary) cut b in M and N respectively. Then, since C is the pole of c', NC is a quadrant. Similarly AM is a quadrant. Therefore $AM + NC = 180°$, or $AN + NM + NM + MC = 180°$. Whence $AC + NM = 180°$. Since B' is the pole of b, MN measures angle B' (art. **8**, th. 1). Therefore $B' + b = 180°$. Similarly $A' + a = 180°$, and $C' + c = 180°$. Because of the reciprocal relation of polar triangles proved in the last theorem, it follows that $A + a' = 180°$, $B + b' = 180°$, and $C + c' = 180°$ also.

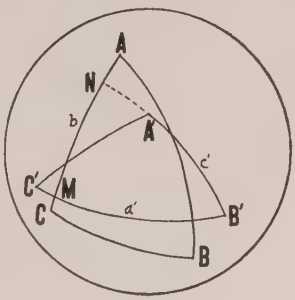

FIG. 4

Since every spherical triangle is the polar triangle of another, this important theorem enables us to formulate, for every theorem concerning the measure of the sides or angles of a spherical triangle, another one concerning the angles and sides.

12. Applications of polar triangle theory. From the theorems of the last article may be derived certain others stated below which are important in the solution of spherical triangles both in finding the unknown parts and in roughly checking the results.

EXERCISES

Using the theorems of article **11,** prove:

Theorem 1. *Each angle of a spherical triangle is less than 180°.*

Theorem 2. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°* (art. **9,** th. 2).

Theorem 3. *If two angles of a spherical triangle are equal so also are the sides opposite, and the triangle is isosceles.*

Theorem 4. *In a birectangular triangle, the sides opposite the right angles are quadrants, and the third side measures the other angle.*

Theorem 5. *An equiangular spherical triangle is also equilateral.*

Theorem 6. *If the sides of a spherical triangle are unequal, that side is greater which is opposite the greater angle.*

Theorem 7. *The sum of two angles of a spherical triangle is less than 180° plus the third angle.*

13. Lunes. Definitions. A **lune** is the portion of the surface of a sphere included between two great semicircles. The **angle** of a lune is the angle formed by the great circles. If two sides of a lune are cut by the arc of a great circle, two triangles are formed which are said to be **co-lunar** with each other.

EXERCISES

1. Show that every spherical triangle has three co-lunar triangles.
2. Prove that the two angles of a lune are equal.

CHAPTER II

THE RIGHT SPHERICAL TRIANGLE

14. Classification according to the number of right angles. In every spherical triangle there are six parts, the three sides and the three angles. We shall see that, *in general*, if three of these elements are known, the remaining three can be found. If one of the parts is a right angle, two other parts suffice for the solution of the triangle.

There are three cases to be considered:

CASE 1. **Three right angles.** It follows from article **12,** theorem 4, that the sides are all quadrants and the triangle is therefore solved.

CASE 2. **Only two right angles.** As above, it follows that the sides opposite the right angles are quadrants. The remaining side, therefore, measures the third angle. This may have any value between 0° and 180°. Unless this third side or the angle opposite is given, the triangle is indeterminate. If one of them is given, the triangle is solved.

CASE 3. **Only one right angle.** Each of the remaining parts may have any value less than 180°, subject to certain restrictions which will appear later.

Since if two or more right angles are given the triangle is either indeterminate or solved, we shall assume that only one right angle is given and, as in plane geometry, denote it by C, and the hypotenuse by c.

15. Formulas for the solution of right spherical triangles. The formulas necessary for the solution of the right spherical triangle are easy to develop.

Let ABC be a right spherical triangle on a sphere whose center is at O. Draw OA, OB, and OC. Then plane BOC is perpendicular

to plane AOC. From P, any point on OA, draw PR perpendicular to OC in plane AOC and RS perpendicular to OB in plane BOC. Then PR is perpendicular to RS (art. **3**, th. 2). Draw PS.

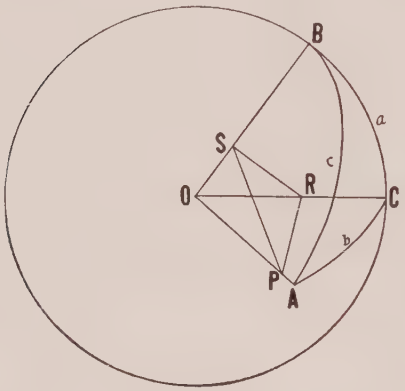

FIG. 5

$$\overline{OP}^2 = \overline{PR}^2 + \overline{OR}^2.$$
$$\overline{OR}^2 = \overline{OS}^2 + \overline{RS}^2.$$
$$\overline{RS}^2 + \overline{PR}^2 = \overline{PS}^2.$$

Adding the three equations gives

$$\overline{OP}^2 = \overline{OS}^2 + \overline{PS}^2.$$

Hence PS is perpendicular to OS. Now

$$\frac{OS}{OP} = \frac{OS}{OR} \cdot \frac{OR}{OP}$$

or $$\cos c = \cos a \cos b. \quad (1)$$

In Figure 5 each side was taken less than a quadrant. What modifications of the above proof are necessary for Figure 6 where the two legs are greater than quadrants, and for Figure 7 where one leg is greater and one less than a quadrant?

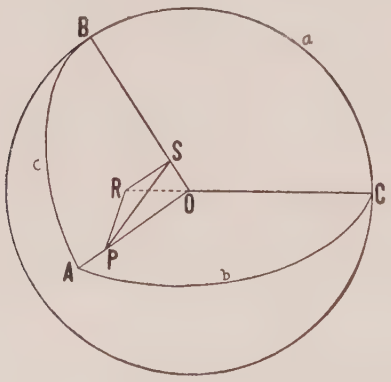

FIG. 6

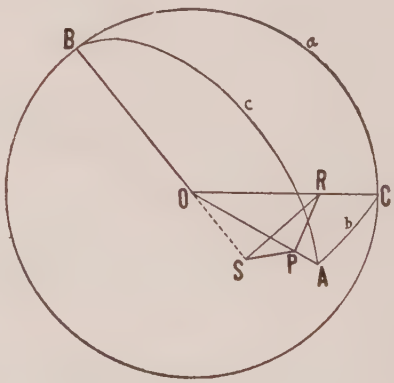

FIG. 7

The last two cases may also be made to depend on the first. When a and b are both greater than quadrants, form the lune CC'. Then formula (1) holds for triangle ABC'. Expressing AC' and BC'

as $180° - b$ and $180° - a$ respectively we get formula (1) for triangle ABC.

When a is greater and b less than a quadrant, construct the lune BB' and proceed as above.

Let the student show with the aid of a figure that the case in which one or more of the sides are quadrants reduces to the case of a birectangular or a trirectangular triangle which has already been discussed.

Since in Figure 5 PS and RS are perpendicular to OB, angle PSR is the measure of the dihedral angle A–OB–C and therefore equals angle B. Using this fact, prove the following identities in a fashion similar to the above:

$$\sin b = \sin B \sin c, \tag{2}$$

$$\tan a = \cos B \tan c, \tag{3}$$

$$\tan b = \tan B \sin a. \tag{4}$$

By interchanging a with b and A with B, we obtain three other forms of the last three formulas.

$$\sin a = \sin A \sin c, \tag{2'}$$

$$\tan b = \cos A \tan c, \tag{3'}$$

$$\tan a = \tan A \sin b. \tag{4'}$$

If we multiply (4) and (4′) together, divide by $\sin a \sin b$, and make use of (1), we get

$$\cos c = \cot A \cot B. \tag{5}$$

Interchanging the members of (2′) and multiplying by (3) gives

$$\sin A \sin c \tan a = \sin a \cos B \tan c.$$

Whence

$$\cos B = \frac{\sin A \cos c}{\cos a}, \text{ or by (1)}$$

$$\cos B = \sin A \cos b, \tag{6}$$

and

$$\cos A = \sin B \cos a. \tag{6'}$$

There are ten different combinations of the parts a, b, c, A, and B taken two at a time. They are represented by these ten equations,

which are sufficient to determine the remaining parts if any two are given. In our construction we have tacitly assumed that no part except the angle C is equal to 90°. The formulas are, however, perfectly general. The student can easily satisfy himself that they are not contradicted if one or more of the other parts are 90°. At most they become indeterminate. For the sake of convenience these formulas and others to follow are assembled at the end of the book. (See Appendix B.)

EXERCISES

1. Show that formulas 2, 3, 4, 2′, 3′, and 4′ hold for Figures 6 and 7.

2. Using the polar triangle theory, write down the ten corresponding formulas for the solution of the quadrantal triangle.

16. About species. Definition. Two parts are said to be of the **same species** if both are greater or both less than 90°.

Certain important conclusions stated below follow from the consideration of the ten formulas of the preceding article. The proof of them is left as an exercise. (*Suggestion:* Using the unit circle, construct the line representations of the trigonometric functions.)

EXERCISES

1. Prove by means of (4) that in a right spherical triangle, a leg and the angle opposite are of the same species.

2. Using (2′) or (6) and exercise 1, show that if A is acute, then $a \leqq A$; and if A is obtuse, then $a \geqq A$; that is, if the angle is acute, the side opposite is no greater, and if the angle is obtuse, the side opposite is no less than the angle.

3. Show by means of (1) that the hypotenuse is less than 90° if, and only if, the other sides are of the same species; that is, in a right triangle either *all* the sides are less than quadrants, or *only one* is less than a quadrant.

4. Show by means of (1) that if a and b are each less than 90°, c is larger than either, and that if a and b are each greater than 90°, c is larger than the supplement of either.

5. Show by means of (1) that c lies between a and $180° - a$.

6. Show by (5) that

(*a*) If A is acute, then B lies between $90° - A$ and $90° + A$.

[*Hint.* Cot A cot B must be numerically less than 1. Hence $|\cot B| < \tan A = |\cot (90° \pm A)|$.]

(*b*) If A is obtuse, then B lies between $90° - (180° - A)$ and $90° + (180 - A)$, that is, between $A - 90°$ and $270° - A$.

7. If $a = A$, the remaining parts are each equal to $90°$.

8. Apply the tests of the preceding exercises to the triangle: $a = 120°$, $b = 17° 47'$, $c = 118° 25'.3$, $A = 100°$, $B = 20° 19'.3$.

These results are useful in determining whether a triangle is possible and something about the unknown parts, as well as in checking roughly one's solutions.

17. Napier's Rules. The ten formulas of article **15** are equivalent to two ingenious rules devised by Napier,* who is already known to the student as the inventor of logarithms. Omitting the right angle, consider the parts of a triangle, ranged on a circle in their natural order. Replace the angles and the hypotenuse by their complements. These five parts are called the **circular parts** of the triangle. Any part may be taken as the middle part. Then the parts next to it are the **adjacent parts,** and the remaining parts are the **opposite parts.** The rules follow.

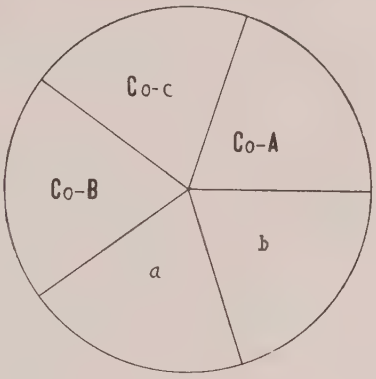

FIG. 8

RULE 1. *The sine of a middle part equals the product of the tangents of the adjacent parts.*

RULE 2. *The sine of a middle part equals the product of the cosines of the opposite parts.*

Since any of the five parts may be taken as the middle part, these rules give rise to ten formulas. Let the student show that they are the formulas of article **15,** thereby proving Napier's rules. They may also be proved independently as Napier himself showed.

18. The six cases of right spherical triangles. We shall now apply the formulas of article **15** to the solution of the right triangle.

* Published in his *Mirifici Logarithmorum Canonis Descriptio* in 1614.

While there are ten different combinations of the parts a, b, c, A, and B, taken two at a time, there are in reality only six different cases as the interchange of a and A with b and B does not give rise to a geometrically distinct case.

CASE 1. **Given the two sides a and b.**

The triangle is always possible and c, A, and B may be determined uniquely from (1), (4), and (4′). Check by (5).

CASE 2. **Given side a and adjacent angle B.**

The triangle is always possible and c, b, and A may be determined uniquely from (3), (4), and (6′). Check by (3′).

CASE 3. **Given the hypotenuse c and side a.**

If c and a are quadrants, the triangle is indeterminate. If not, the triangle is possible only if c lies between a and its supplement (art. **16,** ex. 5). If this condition is satisfied, the triangle is possible and b, A, and B are uniquely determined from (1), (2′), and (3), if it be remembered that a side and angle opposite must be of the same species (art. **16,** ex. 1). Check by (6).

CASE 4. **Given the hypotenuse c and an angle A.**

If both c and A are 90°, the triangle is indeterminate (art. **14,** case 2). Otherwise the triangle is possible and a, b, and B are determined from (2′), (3′), and (5), uniquely, since an angle and side opposite are of the same species. Check by (4).

CASE 5. **Given the two angles A and B.**

The triangle is not always possible. By article **16,** exercise 6, if A is acute, B must lie between $90° - A$ and $90° + A$ and if A is obtuse, B must lie between $A - 90°$ and $270° - A$. If these conditions are satisfied, the triangle is possible and a, b, and c are uniquely determined from (6), (6′), and (5). Check by (1).

CASE 6. **Given side a and the opposite angle A.**

This case corresponds to the ambiguous case in plane trigonometry. The triangle is not always possible. The given parts must be of the same species, the side a being not less than the angle A if A is obtuse, and not greater than A if A is acute (art. **16,** ex. 2). b, c, and B are then determined from (4′), (2′), and (6′). Check by (2).

If $a = A = 90°$, the triangle is indeterminate, for then $c = 90°$ and the remaining parts may have any value.

If $a = A \neq 90°$, the remaining parts are each $90°$ and the triangle is solved.

If $a \neq A$, there will be two solutions since c, b, and B may have either of two supplementary values. To each value of c corresponds only one of the two values of b and B, namely, those which satisfy (1) and (5).

Ex. Discuss these various conditions geometrically with the aid of the lune AA' formed by producing the sides AB and AC until they meet again at A'.

It will be observed that in solving the triangle, those formulas are chosen which employ only the given parts. In checking, the formulas employ only the parts found. Gross errors may sometimes be detected by examining the parts found in the light of the results of article **16.**

Though the logarithmic computation will show when the triangle is impossible, much time may be saved by inspecting the given parts in cases 3, 5, and 6.

EXERCISES

(In this set of exercises and in those which follow the data are assumed as correct to tenths of a minute though for simplicity the zeros which indicate this are omitted.)

Determine which of the following triangles are impossible or indeterminate and why. Solve the remaining triangles (by inspection where possible). Save your results since they will be used as the basis for exercises at the end of Chapter IV.

1. $a = 120°$, $A = 100°$

Solution: Since $a > A$, the triangle is possible and there are two solutions.

$\log \tan a = .23856$	$\log \sin a = 9.93753 - 10$
$\operatorname{colog} \tan A = 9.24632 - 10$	$\operatorname{colog} \sin A = .00665$
$\log \sin b = 9.48488 - 10$	$\log \sin c = 9.94418 - 10$
$b = 17° 47'.0,$	$c = 61° 34'.1,$
or $162° 13'.0.$	or $118° 25'.9.$
$\log \cos A = 9.23967 - 10$	By (1) and (5)
$\operatorname{colog} \cos a = .30103$	*Answers:*
$\log \sin B = 9.54070 - 10$	$b = 17° 47'.0.$ $b' = 162° 13'.0.$
$B = 20° 19'.3,$	$c = 118° 25'.9.$ $c' = 61° 34'.7.$
or $159° 40'.7.$	$B = 20° 19'.3.$ $B' = 159° 40'.7.$

Check:

$\log \sin B = 9.54070 - 10$

$\log \sin c = 9.94418 - 10$

$\log \sin d = 9.48488 - 10$

Note: In the logarithmic work we regarded $\tan A$, $\cos A$, and $\cos a$ as positive. Justify this.

2. $a = 60° 45'.2$, $b = 110° 30'$.
3. $c = B = 90°$.
4. $A = 50° 10'$, $B = 39° 40'$.
5. $b = 95° 31'$, $B = 85° 47'.6$.
6. $c = 35° 22'$, $A = 68° 24'.5$.
7. $a = 79° 27'.7$, $c = 85° 42'$.
8. $A = 120° 36'$, $B = 29° 32'.1$.
9. $a = A = 127° 19'$.
10. $A = 57° 28'$, $B = 45° 33'.2$.
11. $b = 120° 5'$, $B = 105° 38'$.
12. $b = B = 90°$.
13. $b = 77° 16'$, $B = 70° 21'$.
14. $a = c = 90°$.
15. $a = 69° 41'$, $A = 79° 26'.4$.
16. $a = 120° 30'$, $c = 59° 29'$.
17. $b = 127° 15'$, $A = 47° 13'$.

18. Make up two examples under each of cases 3, 5, 6 (giving parts in degrees only) for which the triangle will be (*a*) possible, (*b*) impossible.

19. Figures solvable by means of right spherical triangles. As in plane trigonometry, the solution of regular spherical polygons and isosceles triangles may be accomplished by breaking up the figure into right triangles which may be solved by the methods above.

A quadrantal triangle may be solved by solving the polar triangle, which is rectangular, and taking the supplements of the parts thus found. Or we may solve directly by using the formulas derived in the exercise at the end of article **15.**

Ex. Find a, b, and C if $A = 119° 14'.8$, $B = 69° 30'$, $c = 90°$. The polar of this triangle is the triangle in exercise 2 of the last article. Hence the remaining parts are the supplements of the parts found in that exercise.

CHAPTER III

THE OBLIQUE SPHERICAL TRIANGLE

I. CERTAIN RELATIONS BETWEEN THE TRIGONOMETRIC FUNCTIONS OF THE SIDES AND ANGLES OF A GENERAL SPHERICAL TRIANGLE

20. Law of sines.

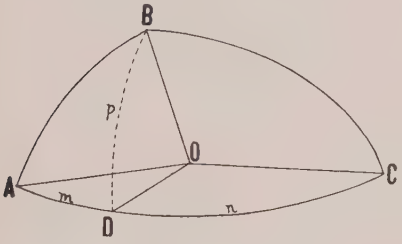

FIG. 9

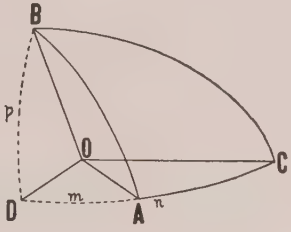

FIG. 10

Let ABC be a spherical triangle. Through B (Fig. 9) pass the arc of a great circle perpendicular to b at D. Then ADB and BDC are right triangles. Denote AD by m, DC by n, and DB by p. It follows from the theory of right triangles that

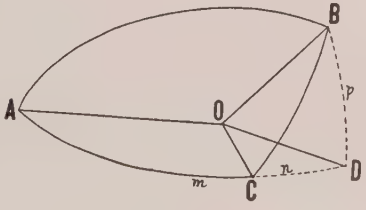

FIG. 11

$$\sin A \sin c = \sin p = \sin C \sin a,$$

whence

$$\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c}.$$

Similarly

$$\frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Therefore

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \tag{1}$$

What modifications of the proof are necessary for Figures 10 and 11 where D falls outside the arc AC?

EXERCISES

1. Compare formula (1) with the law of sines in plane trigonometry.
2. Does the application of the polar triangle theory to (1) give a new formula?
3. Show that if $C = 90°$, (1) reduces to (2) in article **15.**
4. State formula (1) in words.

21. Law of cosines for sides. Using the above construction, it is evident that

$$\frac{\cos a}{\cos n} = \cos p = \frac{\cos c}{\cos m},$$

whence

$$\cos a = \frac{\cos c}{\cos m}\cos (b - m) = \frac{\cos c\ (\cos b \cos m + \sin b \sin m)}{\cos m},$$

that is,

$$\cos a = \cos b \cos c + \sin b \cos c \tan m.$$

But $$\tan m = \cos A \tan c,$$

whence follows the law of cosines for side a:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \qquad (2)$$

A cyclic permutation of the letters gives two other forms of formula (2),

$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

What modifications of the proof are necessary for Figures 10 and 11?

Either the law of cosines for sides or the law of sines may be regarded as fundamental, for either one may be derived first and used to deduce the other. From them follow the formulas for right triangles, and other important formulas.

The law of cosines for sides may be used to prove a set of formulas which are important in astronomy. Multiplying the first formula of (2) by $\cos c$

$$\cos a \cos c = \cos b \cos^2 c + \sin b \sin c \cos c \cos A$$

and subtracting the result from

$$\cos a \cos c + \sin a \sin c \cos B = \cos b$$

gives

$$\sin a \sin c \cos B = \sin^2 c \cos b - \sin b \sin c \cos c \cos A,$$

or, dividing by $\sin c$,

$$\sin a \cos B = \sin c \cos b - \sin b \cos c \cos A, \tag{3}$$

and by cyclic permutation of the letters

$$\sin b \cos C = \sin a \cos c - \sin c \cos a \cos B,$$
$$\sin c \cos A = \sin b \cos a - \sin a \cos b \cos C.$$

if we interchange B and C and therefore b and c, (3) becomes

$$\sin a \cos C = \sin b \cos c - \sin c \cos b \cos A,$$
$$\sin b \cos A = \sin c \cos a - \sin a \cos c \cos B, \tag{4}$$
$$\sin c \cos B = \sin a \cos b - \sin b \cos a \cos C.$$

Ex. Use the polar triangle to derive from (3) and (4) formulas (3′) and (4′) of Appendix C.

EXERCISES

1. Show by means of (2) that a and A are of the same species if a differs from 90° not less than does one of the other sides, that is, if $|\ 90° - a\ | \geqq |\ 90° - b\ |$, or $|\ 90° - a\ | \geqq |\ 90° - c\ |$.

Solving (2) for $\cos A$,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Suppose that $|\ 90° - a\ | \geqq |\ 90° - b\ |$. Then $\cos a$ is numerically at least as great as $\cos b$ and therefore numerically greater than $\cos b \cos c$. Hence the sign of $\cos a$ determines the sign of the numerator, which is also the sign of the fraction since the denominator is always positive. So $\cos A$ will have the same sign as $\cos a$, or a and A will be of the same species.

2. State in words the law of cosines for sides.

3. What does the third formula of (2) become if $C = 90°$?

4. Use exercise 1 to show that in every triangle at least two sides are of the same species as the opposite angles. Does the test tell anything about the remaining parts?

5. Apply the test given in exercise 1 to determine sides which are of the same species as the opposite angles in the triangle $a = 47°$, $b = 55°$, $c = 60°$.

22. Law of cosines for angles. It follows at once from the application of the theory of polar triangles to formula (2), that is, upon replacing each part in the formula by its supplement, that

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a. \tag{2′}$$

EXERCISES

1. Write down the other two forms of (2′).

2. Show that if $C = 90°$, (2′) reduces to (6) of article **15.**

3. If A differs from 90° not less than either B or C does, show that a and A are of the same species.

4. Determine values for the three angles of a triangle so that B and C shall be of the same species as b and c respectively. What do you know of the remaining parts?

23. The trigonometric functions of the half angles in terms of the sides. From (2) and the plane trigonometry formula for the difference of two cosines (see Appendix A).

$$1 - \cos A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \sin c}$$

$$= \frac{2 \sin \frac{(a + b - c)}{2} \sin \frac{(a - b + c)}{2}}{\sin b \sin c}.$$

Let $$a + b + c = 2s.$$

Then $$s - a = \tfrac{1}{2}(b + c - a),$$

$$s - b = \tfrac{1}{2}(a + c - b),$$

and $$s - c = \tfrac{1}{2}(a + b - c).$$

Since the sum of two sides of a triangle is greater than the third side, these quantities are all positive. Then

$$\frac{1 - \cos A}{2} = \frac{\sin (s - c) \sin (s - b)}{\sin b \sin c},$$

or
$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}. \tag{5}$$

Similarly, if we calculate $$\sqrt{\frac{1 + \cos A}{2}},$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}}, \tag{6}$$

whence

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}, \tag{7}$$

or if we define r so that

$$\tan r = \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}},$$

then

$$\tan \tfrac{1}{2} A = \frac{\tan r}{\sin (s - a)}. \tag{8}$$

As in the corresponding formula in plane trigonometry, r is the radius of the inscribed circle.

Since the half angles are each less than 90°, the positive sign must be taken with the radicals.

Each of these formulas except tan r has two other forms found by permuting the letters.

EXERCISES

1. Compare these formulas and their derivation with the corresponding formulas in plane trigonometry.

2. Show that s, $s - a$, $s - b$, and $s - c$ are each less than 180° and hence that the radicands in the above formulas are positive.

24. The trigonometric functions of the half sides in terms of the angles. Let

$$A + B + C = 2 S.$$

By the theory of polar triangles,

$$a' + b' + c' = 540° - (A + B + C),$$

or

$$2 s' = 540° - 2 S,$$

that is,

$$s' = 270° - S,$$

whence

$$\begin{aligned} s' - a' &= (270° - S) - (180° - A) \\ &= 90° - (S - A), \\ s' - b' &= 90° - (S - B), \\ s' - c' &= 90° - (S - C). \end{aligned}$$

Also $$A' = 180° - a \text{ and } \frac{A'}{2} = 90° - \frac{a}{2}.$$

Similarly for B' and C'.

If we apply the formulas of the last article for the cosine, sine, and cotangent of the half angle to the polar triangle $A'B'C'$, make the above substitutions, and simplify, we get

$$\sin \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S - A)}{\sin B \sin C}}, \tag{6'}$$

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}}, \tag{5'}$$

$$\tan \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}}. \tag{9}$$

If we define R so that

$$\tan R = \sqrt{\frac{-\cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}},$$

then

$$\tan \tfrac{1}{2} a = \tan R \cos (S - A). \tag{10}$$

R is the radius of the circle circumscribing the triangle.

There are, of course, two other forms of each of the above formulas except for tan R.

EXERCISES

1. Show that the positive sign should be chosen with each of the radicals in the above formulas.

2. Since $180° < A + B + C < 540°$, show that S is an angle of the second or third quadrant and hence that cos S is negative. Show also that since $S - A = 90° - (s' - a')$, cos $(S - A)$ is positive, as also are cos $(S - B)$ and cos $(S - C)$ and therefore that the radicands in the above formulas are positive.

3. Show that if r' is the radius of the circle inscribed in the polar triangle, $\cot r' = \tan R$.

25. Napier's Analogies.* If the identity

$$\tan \tfrac{1}{2}(a - b) = \frac{\tan \frac{1}{2} a - \tan \frac{1}{2} b}{1 + \tan \frac{1}{2} a \tan \frac{1}{2} b}$$

* First published in the work already cited.

be divided by $\tan \frac{1}{2} c$, and the values of the tangents of the half sides be substituted from (10), the identity will reduce to

$$\frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2} c} = \frac{\cos (S-A) - \cos (S-B)}{\cos (S-C) - \cos S}.$$

Whence by the formula for converting the difference of two cosines into the product of two sines (see Appendix A)

$$\frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2} c} = \frac{-2 \sin \frac{1}{2}(A-B) \sin \frac{1}{2}(2S-A-B)}{-2 \sin \frac{1}{2} C \sin \frac{1}{2}(2S-C)}.$$

But

$$2S - A - B = C \text{ and } 2S - C = A + B,$$

so that

$$\frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2} c} = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}. \tag{11}$$

EXERCISES

1. Write down the other two forms of (11).
2. Express formula (11) in words.
3. Look up in the dictionary this use of the word "analogy."
4. Compare (11) with the Mollweide identity in plane trigonometry,

$$\frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}.$$

5. Prove by the method used in deriving (11) that

$$\frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2} c} = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}. \tag{12}$$

6. Prove directly or from (11) and (12) by means of the polar triangle that

$$\frac{\tan \frac{1}{2}(A-B)}{\cot \frac{1}{2} C} = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \tag{11'}$$

and

$$\frac{\tan \frac{1}{2}(A+B)}{\cot \frac{1}{2} C} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}. \tag{12'}$$

7. Show from (11') that $a - b$ and $A - B$ must have the same sign (which is merely another way of saying that the greater side is opposite the greater angle) and from (12') that $\cos \frac{1}{2}(a+b)$ and $\tan \frac{1}{2}(A+B)$ have the same sign, that is, that $\frac{1}{2}(a+b)$ and $\frac{1}{2}(A+B)$ are of the same species.

26. Delambre's Analogies.* (This article may be omitted if desired.)

If the identity

$$\sin \tfrac{1}{2}(A - B) = \sin \tfrac{1}{2} A \cos \tfrac{1}{2} B - \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B$$

be divided by $\cos \frac{1}{2} C$, the values of the sines and cosines of the half angles be substituted from (5) and (6), and the result simplified by applying the formula for the difference of two sines (see Appendix A), the identity will reduce to

$$\frac{\sin \frac{1}{2}(A - B)}{\cos \frac{1}{2} C} = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2} c}. \tag{13}$$

Similarly it can be proved that

$$\frac{\sin \frac{1}{2}(A + B)}{\cos \frac{1}{2} C} = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2} c} \tag{14}$$

and

$$\frac{\cos \frac{1}{2}(A + B)}{\sin \frac{1}{2} C} = \frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2} c}. \tag{15}$$

If the theory of polar triangles be applied to (13), (14), and (15), (13) and (15) give the same formulas over again, while (14) gives

$$\frac{\cos \frac{1}{2}(A - B)}{\sin \frac{1}{2} C} = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2} c}. \tag{14'}$$

There are, of course, two other forms of each of the above formulas. The formulas of this chapter are collected in Appendix C.

Delambre's Analogies can be derived from Napier's Analogies. The converse is, of course, also true. Both sets can be derived geometrically. The advantage of the above method is that the proofs are based on trigonometric *identities* and hence the size of the various parts does not need to be considered.

It is convenient to avoid negative angles in using these formulas. Since when B is greater than A, b is also greater than a, this may be done by interchanging a with b and A with B.

*Commonly called *Gauss's Formulas*, though incorrectly so, for they were first published by Delambre in the *Connaissance des Temps* for 1809. (See *Philosophical Magazine* 1873. *Note on History of Certain Formulæ in Spherical Trigonometry*, I. Todhunter.)

There are many other interesting and instructive relations which might be considered, but we have now derived all the formulas necessary to solve the general spherical triangle. In fact, Delambre's Analogies are not necessary, but will be found useful in checking.

II. THE SOLUTION OF THE GENERAL SPHERICAL TRIANGLE

27. The six cases of the oblique spherical triangle. The solution of the oblique triangle may be reduced to the solution of the two right triangles formed by dropping an arc of a great circle from one of the vertices, perpendicular to the opposite side. This is a desirable method for the isosceles triangle. In the case of the general triangle, it is, however, better to solve directly by means of the formulas developed in the last section. From these formulas we see that if any three of the parts a, b, c, A, B, C, be given, the remaining three may *in general* be found.

There are twenty different combinations of these six parts taken three at a time, but only six of these combinations differ from each other geometrically, and three of the six cases may be made to depend on the other three by the theory of polar triangles. The six cases in pairs are as follows:

1. Given the three sides.
1′. Given the three angles.

2. Given two sides and the included angle.
2′. Given two angles and the included side.

3. Given two sides and the angle opposite one of them.
3′. Given two angles and the side opposite one of them.

With the exception of 1′, the analogy between these cases and the corresponding cases in plane geometry is apparent.

In Cases 1′, 2′, and 3′, the polar triangle may be solved under Cases 1, 2, and 3 respectively. The supplements of the parts thus found will be the desired solution, or the triangles may be solved directly.

28. CASE 1. **Given the three sides.**

If the sum of two sides is greater than the third, and the sum of all the sides is less than 360°, the triangle is possible, and uniquely determined.

If only one angle is desired, it may be found by using (5), (6), or (8). If the complete solution is desired, (8) is preferable.

While any of Napier's or Delambre's Analogies which involve the three angles may be used for checking, perhaps the law of sines involves less labor and offers fewer chances for errors in computation. The angles may also be found by (2), but this formula is not adapted to logarithmic computation.

CASE 1′. **Given the three angles.**

If the sum of the angles is greater than 180° and less than 540°, and if the sum of two angles is less than 180° plus the third (art. **12,** th. 7), the triangle is possible and uniquely determined. Solve by (5′), (6′), or preferably (10), or (without the use of logarithms) by (2′). Check by any of Napier's or Delambre's Analogies involving the three sides, or more easily by the law of sines.

29. CASE 2. **Given two sides and the included angle.**

The triangle is always possible and uniquely determined. Use (11′) and (12′) to find one half the sum and one half the difference of the other two angles and from these, the angles themselves. If we use the law of sines to find the third side, since the side is determined from its sine, it may not be evident which of the two supplementary values to choose. This may *sometimes* be decided by remembering that the greater side is opposite the greater angle. But to avoid this complication use (11) or (12) to find the third side. Here no ambiguity arises. Check by any of Delambre's Analogies or by the law of sines.

CASE 2′. **Given two angles and the included side.**

The triangle is always possible and uniquely determined. Find the two sides from (11) and (12) and the remaining angle from (11′) or (12′). Check by the law of sines or by any of Delambre's Analogies.

30. CASE 3. **Given two sides and the angle opposite one of them.**

This corresponds to the ambiguous case in plane trigonometry, and, here also there may be two solutions, one, or none at all. Suppose a, b, and A are given. Then B may be found from the law of sines, and c and C from Napier's Analogies. Check by any of Delambre's Analogies.

Since B is determined from $\sin B = \dfrac{\sin b \sin A}{\sin a}$ there will be no solution if the numerator is greater than the denominator, that is, if $\log \sin B > 0$; one solution, and that a right triangle, if $\sin b \sin A = \sin a$, that is, if $\log \sin B = 0$; and a possibility of two solutions if $\sin b \sin A < \sin a$, that is, if $\log \sin B < 0$. In this case it will be possible to find two supplementary values of B, but only values of B which satisfy the condition that $A - B$ and $a - b$ have the same sign are permissible (see art. **25,** ex. 7). It is possible that both values of B will satisfy this condition, or only one, or neither. Hence there may be two solutions, one, or none. If the condition is satisfied, the values of $\tan \frac{1}{2}c$ and $\cot \frac{1}{2}C$ in (11) and (11′) will be positive and we can find c and C. (In the two-solution case there will, of course, also be two values of c and of C.)

CASE 3′. **Given two angles and the side opposite one of them.**

This case is also ambiguous. As in Case 3, the triangle may be impossible, or it may have one or two solutions. To solve, use the law of sines to find a second side and Napier's Analogies for the remaining parts. Check by any of Delambre's Analogies.

Suppose that A, B, and a are given. There will be no solution if $\sin B \sin a > \sin A$; one solution, and that a quadrantal triangle, if $\sin B \sin a = \sin A$; and if $\sin B \sin a < \sin A$ two solutions, one, or none, according as each of the supplementary values of b, one, or neither satisfies the condition that $a - b$ shall have the same sign as $A - B$.

31. Tables for determining the number of solutions in the ambiguous cases. The geometric discussion of the ambiguous cases, while simple, is rather long. If $\sin b \sin A > \sin a$, a is shorter

than the perpendicular arc dropped from C on side c. If $\sin b \sin A = \sin a$, a is equal to this perpendicular, and the triangle is right-angled at B. If $\sin b \sin A < \sin a$, the geometric considerations are summarized in the following tables.

TABLE 1. a EQUAL TO NEITHER b NOR THE SUPPLEMENT OF b

RELATION OF a TO b AND ITS SUPPLEMENT	NUMBER OF SOLUTIONS IF $A < 90°$	NUMBER OF SOLUTIONS IF $A > 90°$
a less than both	2	0
a between the two	1	1
a greater than both	0	2

TABLE 2. a EQUAL TO b OR THE SUPPLEMENT OF b

RANGE OF a	NUMBER OF SOLUTIONS IF $A < 90°$	NUMBER OF SOLUTIONS IF $A > 90°$
$a < 90°$	1	0
$a = 90°$	0	0
$a > 90°$	0	1

We will prove one of the rules given above as an illustration of the method. It is suggested that the student prove several of the others. Let us prove (Table 1) that if a is greater than both b and its supplement and A is obtuse there are two solutions. Construct the sines of a, b, and $180° - b$ by means of the unit circle. Then since a is greater than either b or $180° - b$, $\sin a < \sin b$. But $\sin B = \dfrac{\sin b}{\sin a} \sin A$. Therefore $\sin B > \sin A$, or $A > B$. Since $\sin (180° - B) = \sin B$, $\sin (180° - B) > \sin A$ also. Hence $A > 180° - B$, that is, the condition that A is greater than either of the supplementary values of B found from the formula is satisfied, and there are two solutions.

It would be well to illustrate these various cases by constructing angle A on a ball and completing the lune AA'. Consider the cases when A is acute and when A is obtuse for the different relations of a and b mentioned in the tables.

By applying the polar triangle theory to the two tables under the last case, that is, by replacing a, b, and A by $180° - A$, $180° - B$, and $180° - a$, respectively, we get two other tables giving the geometric conditions governing the number of solutions for case $3'$ when $\sin B \sin a < \sin A$.

TABLE 1′. A EQUAL TO NEITHER B NOR THE SUPPLEMENT OF B

RELATION OF A TO B AND ITS SUPPLEMENT	NUMBER OF SOLUTIONS IF $a > 90°$	NUMBER OF SOLUTIONS IF $a < 90°$
A greater than both	2	0
A between the two	1	1
A less than both	0	2

TABLE 2′. A EQUAL TO B OR THE SUPPLEMENT OF B

RANGE OF A	NUMBER OF SOLUTIONS IF $a > 90°$	NUMBER OF SOLUTIONS IF $a < 90°$
$A > 90°$	1	0
$A = 90°$	0	0
$A < 90°$	0	1

It will be observed that Tables 1′ and 2′ can be obtained from Tables 1 and 2 by replacing a, b, and A by A, B, and a respectively and the words "greater than" by "less than," and vice versa.

Tables 1 and 2 are readily adapted to general use if a, b, and A are replaced respectively by the words *opposite side*, *adjacent side*, and *angle*. But the tables stand out more clearly if a concrete example is taken as above.

A corresponding substitution may be made in Tables 1′ and 2′.

32. Suggestions for checking solutions. Before starting the logarithmic solution of any triangle, determine whether the data are permissible. Then make out a complete form for the solution including the check before looking up any logarithms. For example, suppose a, b, and C are given.

$$\begin{aligned} a &= \\ b &= \\ a-b &= \\ a+b &= \end{aligned} \qquad \begin{aligned} \tfrac{1}{2}(a-b) &= \\ \tfrac{1}{2}(a+b) &= \\ \text{Check: } a &= \\ b &= \end{aligned}$$

$$\begin{aligned} (9')\ \log \sin \tfrac{1}{2}(a-b) &= \\ \operatorname{colog} \sin \tfrac{1}{2}(a+b) &= \\ \log \cot \tfrac{1}{2} C &= \\ \hline \end{aligned} \qquad \begin{aligned} (10')\ \log \cos \tfrac{1}{2}(a-b) &= \\ \operatorname{colog} \cos \tfrac{1}{2}(a+b) &= \\ \log \cot \tfrac{1}{2} C &= \\ \hline \end{aligned}$$

$$\begin{aligned} \log \tan \tfrac{1}{2}(A-B) &= \\ \tfrac{1}{2}(A-B) &= \\ \tfrac{1}{2}(A+B) &= \\ A &= \\ B &= \end{aligned} \qquad \begin{aligned} \log \tan \tfrac{1}{2}(A+B) &= \\ \tfrac{1}{2}(A+B) &= \end{aligned}$$

$$\begin{aligned} (12)\ \operatorname{colog} \sin \tfrac{1}{2}(A+B) &= \\ \log \cos \tfrac{1}{2}(a-b) &= \\ \log \cos \tfrac{1}{2} C &= \\ \log \cos \tfrac{1}{2} c &= \\ \hline \tfrac{1}{2} c &= \\ c &= \end{aligned}$$

Check: $\sin \tfrac{1}{2}(A-B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a-b) \cos \tfrac{1}{2} C$

$$\begin{aligned} (13)\ \log \sin \tfrac{1}{2}(A-B) &= \\ \log \sin \tfrac{1}{2} c &= \\ \hline \end{aligned} \qquad \begin{aligned} \log \sin \tfrac{1}{2}(a-b) &= \\ \log \cos \tfrac{1}{2} C &= \\ \hline \end{aligned}$$

Gross errors may sometimes be detected by examining the solution in the light of the following principles:

The sum of the sides is less than 360°.

The greater side is opposite the greater angle.

The difference between any two sides has the same sign as the difference between the two angles opposite taken in the same order.

A side which differs from 90° not less than does another side is of the same species as the angle opposite.

At least two of the sides are of the same species as the angles opposite.

EXERCISES

Determine which of the following triangles are impossible and solve the others:

[NOTE: Save all results, as these data will be used as a basis for the examples in the following chapter.]

1.	$C = 165° 24'$,	$a = 48°$,	$b = 69° 48'$.
2.	$a = 115° 20'$,	$b = 45° 6'$,	$c = 75° 12'$.
3.	$b = 31° 48'$,	$c = 55° 52'$,	$C = 102° 31'$.
4.	$A = 91°$,	$B = 62°$,	$C = 23°$.
5.	$A = 53° 13'$,	$B = 96° 50'$,	$C = 31° 18'$.
6.	$A = 108° 10'$,	$C = 79°$,	$b = 69° 20'$.
7.	$A = 138°$,	$C = 130°$,	$a = 147° 36'$.
8.	$a = 100° 17'$,	$b = 60° 19'$,	$c = 75°$.
9.	$b = 95°$,	$B = 47°$,	$C = 59°$.
10.	$B = 35° 37'$,	$b = 30° 19'$,	$c = 47° 30'$.
11.	$A = 167° 12'$,	$B = 96° 48'$,	$C = 31° 20'$.
12.	$A = 75°$,	$a = 36° 25'$,	$b = 143° 35'$.
13.	$a = 17° 26'$,	$c = 55°$,	$C = 70°$.
14.	$B = 138° 29'$,	$b = 76° 28'$,	$c = 76° 28'$.
15.	$B = 119° 3'$,	$C = 99° 9'$,	$a = 87° 39'$.
16.	$C = 126° 19'$,	$a = 65°$,	$c = 142° 15'$.
17.	$a = 36°$,	$b = 59°$,	$c = 84°$.
18.	$C = 102°$,	$B = 78°$,	$c = 100°$.
19.	$a = 109° 10'$,	$b = 121°$,	$c = 130° 6'$.
20.	$A = 100°$,	$B = 70°$,	$C = 91°$.
21.	$a = 100° 10'$,	$b = 45° 16'$,	$c = 54° 3'$.
22.	$a = b = 90°$,	$A = 45°$.	
23.	$b = c = 120°$,	$A = 55°$.	
24.	$A = 48°$,	$C = 40°$,	$a = 100°$.
25.	$A = 80°$,	$B = 70°$,	$b = 65°$.
26.	$A = B = 37°$,	$b = 93°$.	

CHAPTER IV

THE AREA OF A SPHERICAL TRIANGLE

33. Introductory theorems concerning areas. Definitions. Two triangles are **symmetrical** if their corresponding parts are equal, but placed in reverse order. The triangle $A'B'C'$ formed by the opposite extremities of the diameters drawn through A, B, C is therefore symmetrical to ABC. Triangles ABC and $A'B'C'$ are called **vertical** triangles.

The following theorems are stated without proof:

Theorem 1. *Two triangles on the same sphere or equal spheres whose corresponding sides are equal are either congruent or symmetrical.*

Theorem 2. *Two symmetrical triangles on the same sphere or equal spheres have the same area.*

Theorem 3. *The surface of a sphere is equal to the area of four great circles,* or $S = 4\pi R^2$.

Theorem 4. *The areas of two lunes on the same sphere or equal spheres are in the same ratio as their angles.*

34. Spherical degrees. Let the circumference of a great circle be divided into 360 parts, each equal to one degree. Through the points of division and the axis of the circle pass planes. The surface of the sphere is divided in this way into 720 equal biquadrantal triangles, each one of which is called a **spherical degree.** Each of the lunes equals two spherical degrees. Hence if L be the area of a lune whose angle is A, by theorem 4 of article **33,**

$$L : 2 = A : 1$$

or

Theorem. *The area of a lune in spherical degrees is twice the number of degrees in the angle of the lune.*

The ratio of the area of any spherical figure to the area of the sphere is, of course, the same whether expressed in spherical degrees, or in square units of radius. So if S' is the area in spherical degrees, and S in square measure, then

$$S' : 720 = S : 4\pi R^2. \tag{1}$$

This relation enables one to translate the area from one measure into the other.

35. The area of the spherical triangle in spherical degrees and in square units of radius. Definition. The **spherical excess** E of a triangle is the difference between the number of degrees in the sum of the angles of the triangle and 180, that is, if A, B, and C represent the *numerical measure* of the angles, $E = A + B + C - 180$.

Theorem: *The area of a spherical triangle expressed in spherical degrees is equal to the spherical excess of the triangle.*

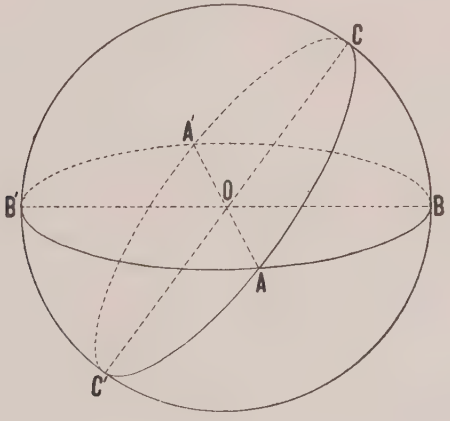

FIG. 12

Let ABC be a spherical triangle. Produce the sides to form the lunes AA', BB', CC'. Then

$$ABC + A'BC = \text{lune } ABA'C, \text{ or lune } A,$$
$$BAC + B'AC = \text{lune } BAB'C, \text{ or lune } B,$$
$$CAB + C'AB = \text{lune } CBC'A, \text{ or lune } C.$$

But by article **33,** theorem 2, $A'CB$ may be replaced by its vertical triangle $AB'C'$ in the first equation. Hence by addition

$$2\,ABC + (ABC + AB'C' + B'AC + C'AB)$$
$$= \text{lune } A + \text{lune } B + \text{lune } C.$$

The sum of the four triangles in parenthesis is a hemisphere, or 360 spherical degrees. So that

$$2\,ABC = (2\,A + 2\,B + 2\,C - 360) \text{ spherical degrees, or}$$
$$ABC = (A + B + C - 180) \text{ spherical degrees}$$
$$= E \text{ spherical degrees}$$

or
$$ABC = E. \tag{2}$$

This area is independent of the size of the sphere. The area in square units is, however, a function of the radius. Substituting E for S' in (1) and representing by E' the number of radians in E degrees we have

$$\text{area } ABC = \frac{\pi}{180} ER^2 = E'R^2. \tag{3}$$

That is, *the area of a triangle in square units of radius is equal to the product of the spherical excess expressed in radians by the square of the radius.*

Ex. Find the area on the earth's surface of the triangle whose angles are 60° 20′, 40°, 85°, taking R as 3960 miles.

Solution: Here $E = 5°\ 20'$. Hence $E' = .09307$.

$$\log E' = 8.96881 - 10$$
$$2 \log R = 7.19540$$
$$\log S = 16.16421 - 10$$
$$S = 1{,}459{,}500 \text{ square miles.}$$

36. L'Huillier's formula for the area in terms of the sides. It is sometimes desirable to find the area, that is, to find E, without first solving the triangle. There are a number of formulas for E in terms of the various parts of the triangle. As an illustration, we mention one, L'Huillier's formula in terms of the three sides, namely,

$$\tan \tfrac{1}{4} E° = \sqrt{\tan \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b) \tan \tfrac{1}{2}(s - c) \tan \tfrac{1}{2} s}.$$

For

$$\tan \tfrac{1}{4} E^\circ = \tan \tfrac{1}{4}(A + B + C - 180^\circ)$$
$$= \tan \tfrac{1}{2}\left[\left(\frac{A+B}{2}\right) - \left(90^\circ - \frac{C}{2}\right)\right].$$

But from plane trigonometry (see Appendix A), we know that

$$\tan \tfrac{1}{2}(x - y) = \frac{\sin x - \sin y}{\cos x + \cos y},$$

so that

$$\tan \tfrac{1}{4} E^\circ = \frac{\sin \frac{1}{2}(A+B) - \cos \frac{1}{2} C}{\cos \frac{1}{2}(A+B) + \sin \frac{1}{2} C}$$
$$= \frac{\cos \frac{1}{2}(a-b) - \cos \frac{1}{2} c}{\cos \frac{1}{2}(a+b) + \cos \frac{1}{2} c} \cdot \frac{\cos \frac{1}{2} C}{\sin \frac{1}{2} C} \quad \text{[art. 26, (14), (15)]}$$
$$= \frac{\sin \frac{1}{4}(a-b+c) \sin \frac{1}{4}(b+c-a)}{\cos \frac{1}{4}(a+b+c) \cos \frac{1}{4}(a+b-c)}$$
$$\times \sqrt{\frac{\sin s \sin (s-c)}{\sin (s-a) \sin (s-b)}} \quad \text{[art. 23, (5), (6)]}$$
$$= \frac{\sin \frac{1}{2}(s-b) \sin \frac{1}{2}(s-a)}{\cos \frac{1}{2} s \cos \frac{1}{2}(s-c)}$$
$$\times \sqrt{\frac{4 \sin \frac{1}{2} s \cos \frac{1}{2} s \sin \frac{1}{2}(s-c) \cos \frac{1}{2}(s-c)}{4 \sin \frac{1}{2}(s-a) \cos \frac{1}{2}(s-a) \sin \frac{1}{2}(s-b) \cos \frac{1}{2}(s-b)}}.$$

Whence

$$\tan \tfrac{1}{4} E^\circ = \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)}. \quad (4)$$

Formulas (2) and (4) enable us to get the area when the three angles or the three sides of a spherical triangle are given. In the other cases we may solve for the remaining sides or angles. Other formulas may be found by referring to the longer treatises on spherical trigonometry.

Ex. Find the areas in spherical degrees and in square measure of some of the triangles in the exercises at the end of Chapters II and III, supposing them to be on a sphere whose radius is 10 centimeters.

CHAPTER V

PRACTICAL APPLICATIONS OF SPHERICAL TRIGONOMETRY

Spherical trigonometry finds its most important applications in the fields of geography, geodesy, and astronomy (including, of course, navigation). Indeed, it was first developed to meet the needs of the astronomers.

I. APPLICATIONS TO GEOGRAPHY

37. Latitude and longitude. For most practical purposes, the earth may be regarded as a sphere whose radius is 3960 miles. An arc of one degree is 60 geographical miles. The diameter around which the earth revolves daily is called its **axis** and its extremities are the **north** and **south poles.** The great circle perpendicular to the axis is the **equator** and the great circles through the axis perpendicular to the equator are **meridians.** The **prime meridian** passes through the observatory at Greenwich, England. Circles whose planes are parallel to the plane of the equator are **parallels of latitude.** The **longitude** of a place is the angle at the pole which the meridian of the place makes with the prime meridian. It is measured by an arc of the equator east or west from the prime meridian. It is therefore never more than 180°. The **latitude** ϕ of a place is its angular distance measured from the equator north or south along the meridian of the place, and is not more than 90°. Instead of using negative numbers we speak of *east* and *west* longitudes, or *north* and *south* latitudes. The position of a point on the earth's surface is known when its latitude and longitude are known.

38. Great circle sailing. If it be desired to sail from A to B by the shortest route, one must follow the arc of a great circle. The acute angle between this arc and the meridian of A is the **bearing** of B from A and would differ from the bearing of A from B. A

ship sailing on the arc of a great circle must always be changing its course. For example, in Figure 13 the bearing of B from A is N. 45° E., while the bearing of A from B is S. 50° W.

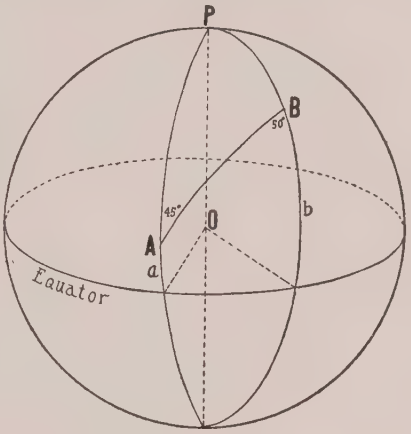

FIG. 13

EXERCISES

(For the latitudes and longitudes of places mentioned in the following examples see Appendix E. In problems relating to aviation, the distance of the airship above the earth's surface is neglected.)

1. To find the distance between two points when their latitudes and the difference of their longitudes are known. Let A and B be the two places, a and b their latitudes, and d the difference between their longitudes. Then in the triangle PAB, $AP = 90° - a$, $BP = 90° - b$, and the included angle $APB = d$ are known. Hence the triangle may be solved by the methods of Chapter III. In case of south latitudes replace a and b by their negatives.

2. When will the bearing of each of two places from the other be numerically the same? Would a ship sailing from one place to the other on the arc of a great circle need to change its course in this case?

3. Show that if d is the number of degrees in the arc of a great circle on the earth's surface and m its length that $d = \frac{m}{22\pi}$. In May 1926, the *Norge* flew from Spitzbergen to Point Barrow over the North Pole. How far did it travel, assuming that it followed the arcs of great circles?

4. A ship bears west from Honolulu. How far has it sailed when it reaches a point whose longitude is 180°? What is the latitude of the place and the bearing of Honolulu from it?

5. The bearing of A (75°E.) from B (45° W.) is N. 65° E. and the bearing of B from A is S. 80° W. Find the latitudes of both places and the distance between them.

6. The latitude of a point A is 40° 43′ N. and the latitude of B is 53° 24′ N. A is 51° 20′ from B. Find the difference of their longitudes and the bearing of each place from the other, B being farther east.

7. A point A (50° N., 50° E.) is 3000 miles from a place B (8° W.). Find the latitude of B and its bearing from A.

8. The bearing of Rio de Janeiro from London is S. 39° 22′.2 W. The bearing of London from Rio de Janeiro is N. 25° 22′.4 E. Regarding the latitude and longitude of London as known, find the latitude and longitude of Rio de Janeiro and the distance between them in miles.

9. A ship, latitude 25° 10′ S., starts on a course N. 45° W. What course will it be steering after following the arc of a great circle for 900 miles?

10. A ship sails S. 75° W. from San Francisco. What distance will it have sailed when its course is S. 81° 3′ W.? What are its latitude and longitude then?

11. A ship sails from Sandy Hook to Gibraltar by the shortest route. What is its bearing at the start and what distance does it travel?

12. Show that the radius r of a parallel of latitude ϕ is $r = 3960 \cos \phi$ miles in length.

13. What is the length of the parallel of latitude $\phi = 60°$?

14. How far is New York from the earth's axis? How far does the city travel in the daily rotation of the earth about its axis?

15. The shortest distance between two places on the same parallel of latitude $\phi = 60°$ is 600 miles. If the distance were measured along the parallel of latitude, how much greater would it be?

16. An airship flies from London to Pernambuco along the arc of a great circle at the rate of 100 miles per hour. How long does it take to make the journey?

17. Find the distance from Liverpool to New York.

18. Find the distance from the place where you live to Tokio.

(In the following problems assume that the course followed was the arc of a great circle.)

19. In May 1926 Byrd and Bennett flew from Spitzbergen to the North Pole and return. If their average speed was 109 miles per hour how long did it take them?

20. In May 1927 Lindbergh made the first non-stop flight from New York to Paris in $33\frac{1}{2}$ hours. Find his average speed per hour.

21. In June 1927 Chamberlin and Levine flew from New York to Eisleben, 110 miles from Berlin. What was their direction when they took off from New York?

22. In June 1927 Hegenberger and Maitland broke the world's record for distance over water in their flight from San Francisco to Honolulu in $26\frac{1}{2}$ hours. Find the distance flown and their average speed.

II. APPLICATIONS TO ASTRONOMY

39. The celestial sphere. Let us suppose the heavenly bodies projected from the eye of the observer on the inner surface of a celestial sphere. Since even the nearest star is so far away from

the earth, the radius of this sphere is very great and the size of the earth so small by comparison as to be negligible.*

The position of a star is determined when its coördinates with respect to some system are known. Though there are five systems in use, the ecliptic and galactic systems are used comparatively little and will not be considered in this book. In each system the reference lines are mutually perpendicular great circles. Various notations for the coördinates of the different systems are used. The writer has adopted the notation of W. W. Campbell in his *Elements of Practical Astronomy*.

40. The horizon system. The plane tangent to the earth through the feet of the observer cuts the celestial sphere in his **horizon.** However, since the radius of the earth is so small compared with the radius of the celestial sphere, a plane through the center of the earth parallel to the tangent plane would appear to cut the celestial sphere in the horizon also. The perpendicular to this plane, or **plumb-line,** cuts the celestial sphere above in the **zenith** and below in the **nadir. Vertical circles** are great circles passing through the zenith and nadir, and are perpendicular to the horizon. The vertical circle passing through the north pole is the **meridian** of the observer. The meridian and horizon are the reference circles for this system.† The coördinates are the *azimuth* ‡ (= **A**) and the *altitude* (= **h**). The **azimuth** is the angle between the meridian and the vertical circle passing through the star, and is usually measured by an arc of horizon *from the south point west* from 0° to 360°.§ **Altitude**

* *Alpha Centauri*, the nearest star system of the first magnitude in the southern hemisphere, is 275,000 times as far away from the earth as the sun, which is 93,000,000 miles away, a figure which transcends our powers of imagination. In order to get some comprehension of this distance we translate it into terms of light. The light from this star takes $4\frac{1}{3}$ years to reach us, traveling 186,000 miles per second. In other words, we see not the star of today but the star of over four years ago. The large majority of stars found on photographs of the heavens are so far away that the light by which their presence is known must have been on its way to us during the whole of human history.

† The **prime vertical** passing through the east and west points is sometimes used as a standard.

‡ The word is Arabic in origin and has the same meaning as **true bearing** in surveying and navigation.

§ Surveyors sometimes measure from the north point.

is the angular distance of the star from the horizon. It is measured along the vertical circle of the star from $-90°$ to $+90°$, being negative for points below the horizon. Thus to find a star whose azimuth is 135° and whose altitude is $\pm$ 45° we face northwest and look halfway up to the zenith. Sometimes the distance of the star from the zenith is used instead of the altitude. **Zenith distance** ($= z$) varies from 0° to 180°. In Figure 14, O is the earth, $NESW$ is the horizon, Z the zenith, P the north pole, NPZ the meridian, B the star, HBZ its vertical circle, $\angle SZH$ or arc $SWNH$ its azimuth, HB the altitude, and ZB the zenith distance. It will be clear from the next article that the azimuth and altitude depend both upon the position of the observer and upon the time of observation.

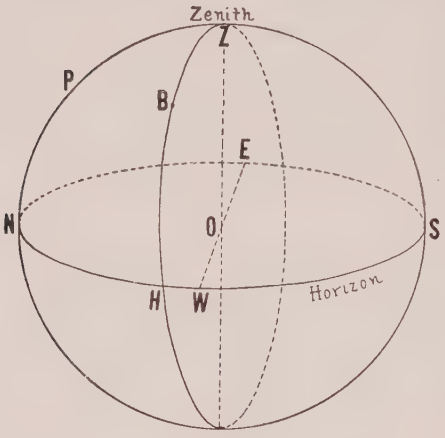

FIG. 14

41. First equatorial system. Let the axis of the earth and the planes of its equator and meridians be extended indefinitely. They will cut the celestial sphere in the **poles, celestial equator,** and **hour circles** respectively. The **celestial meridian** of the observer is the hour circle cut out by the plane of his terrestrial meridian and passes through both poles and the zenith. The meridian and celestial equator are the circles of reference for this system.

Since the earth rotates daily about its axis, every point on the celestial meridian will generate a circle every twenty-four hours at the rate of 15° per hour. So angles are measured in time. 15° = 1 hour. Instead of thinking of the earth as moving, we may think of the stars as moving from east to west and completing a revolution daily about the celestial polar axis. The **hour angle** ($= t$) of a star is the angle between the celestial meridian and the hour circle passing through the star. It is measured (in north

latitudes) *from the meridian*, or south point of the equator, around *toward the west*, from 0° to 360°, or from 0 hours to 24 hours.* The **declination** (= **δ**) of a star is its angular distance from the celestial equator, measured along the hour circle of the star, and is exactly analogous to the terrestrial latitude of a place. It varies from − 90° to + 90° being negative for points south of the equator. Sometimes the **north polar distance** (= **P**) is used instead. This distance varies from 0° to 180°.

Since the celestial meridian is different for observers in different places and since the star is (apparently) continually moving, the hour angle of a star depends upon the place of observation while the declination remains always the same. The declination of the *sun*, however, varies from 23° 27′ S. to 23° 27′ N., due to the annual motion of the earth about the sun and to the fact that the plane of the earth's orbit is not parallel to the plane of the equator (see art. **42**).

The declination of the sun is given for every day in the year in the *Nautical Almanac*.

In Figure 15, P is the north pole, $ERQW$ the equator, Z the zenith, NB_2S the horizon, $B_0B_1B_2$ the circle described by a star in its daily course about the axis. At B_0 the hour angle of the star is 0° and its declination is QB_0. After twelve hours the star is at B_1. Its hour angle is now 180°. At B_2 the star rises above the horizon. Its hour angle is 180° + angle NPR. It gets back to B_0 again after twenty-four hours.

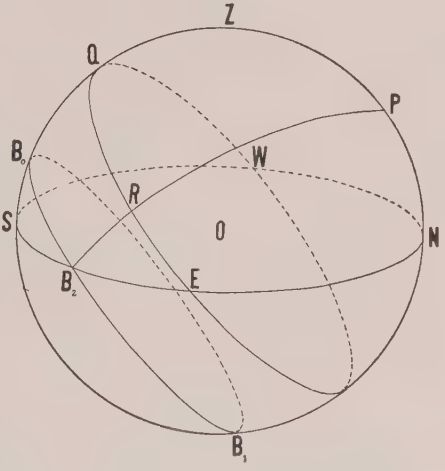

FIG. 15

42. The ecliptic. The earth revolves about the sun annually. The plane of its orbit cuts the celestial sphere in the **ecliptic,** which

* It is measured from the *north* point west in *south* latitude.

may also be defined as the path of the sun in its apparent annual motion. This path cuts the celestial equator in two points, the **equinoxes,** so called since on these dates the days equal the nights in length. The sun goes through the **vernal equinox** in crossing from the south to the north side of the equator. This happens on or within one day of March 21. The other crossing, about September 23, is the **autumnal equinox.** The plane of the earth's orbit makes an angle of approximately $23° 27'$ with the plane of the equator. This angle is called the **obliquity** of the ecliptic. In Figure 16, CVL is the ecliptic and M its pole.

43. A second equatorial system. Instead of using the declination and hour angle, we may use the declination and distance intercepted on the equator between the vernal equinox and the hour circle of the star. This distance is called the **right ascension** ($= \alpha$) and is measured *from the vernal equinox toward the east* through 24 hours or 360°. The hour circle of the equinoxes is called the **equinoctial colure.** In Figure 16, P is the pole, EVR the equator, B the star, VR (or angle VPR) its right ascension, and RB its declination. Right ascension like declination is independent of the observer's position and of the time of day.

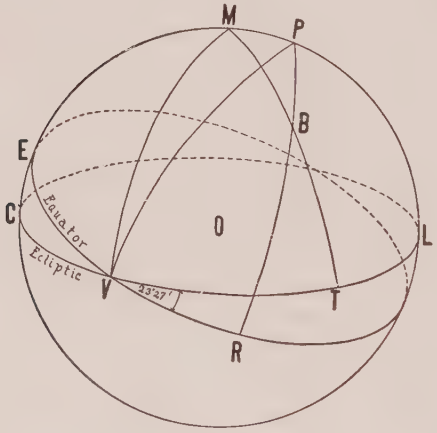

FIG. 16

44. Comparison of the three systems of coördinates.* The horizon system depends upon the position of the observer and his plumb line. Each place and each hour has its own zenith, horizon, and meridian. The stars have a daily motion across the sky and cross the meridian at different times on different days. Hence

* See F. R. Moulton's *Introduction to Astronomy*, p. 127.

azimuth and altitude alone are not enough to locate a body. The place of observation, the time of day, and the day of the year must all be given.

In the first equatorial system the hour angle depends upon the longitude of the observer, the time of day, and the day of the year, while the declination of a star is independent of all of these.

Except for very slow changes which we may disregard here, the coördinates of the second equatorial system are independent of the place of observation, the time of day, and the day of the year. With this understanding, the apparent position of a star is defined when its right ascension and declination are known. Hence this system is convenient for cataloguing the stars.

We summarize the three systems of coördinates in the following table.

Defined by	System	Reference Circles		Coördinates		Origin
		Primary	Secondary	Primary	Secondary	
Direction of gravity	I	horizon	meridian of observer	azimuth $= A$	altitude $= h$	south point
Rotation of earth	II	equator	meridian of observer	hour angle $= t$	declination $= \delta$	foot of meridian on equator
Rotation of earth	III	equator	equinoctial colure	right ascension $= \alpha$	declination $= \delta$	vernal equinox

45. Projection of three systems on the plane of the horizon. Perhaps the most difficult thing for the beginner to understand is the *direction* in which the primary coördinates are measured. In Systems I and II (in north latitudes) they are measured *from the origin of coördinates west.* In System III they are measured *from the vernal equinox east.* The whole matter will be clearer if we project the celestial sphere by lines parallel to the plumb line on the plane of the horizon. The horizon circle is its own projection. The zenith projects into the center of this circle and all vertical circles into diameters. The ecliptic, equator, and arcs of great circles not

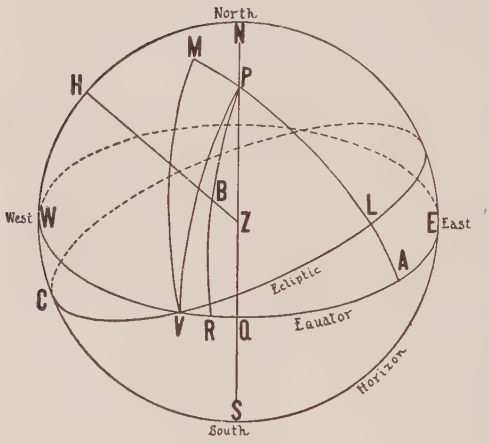

FIG. 17

through the zenith will project into ellipses and arcs of ellipses. In Figure 17, the four systems are represented, the lettering being the same as in the preceding figures. $NESW$ is the horizon, Z the zenith, WVE the equator, P the north pole, CVL the ecliptic, M its pole, B the star, and $AL = PM$ the obliquity of the ecliptic.

System I. HB = altitude = h.
SH or $\angle SZH$ = azimuth = A.

System II. QR or $\angle QPR$ = hour angle = t.
RB = declination = δ.

System III. VR or $\angle VPR$ = right ascension = α.
RB = declination = δ.

There are, of course, equations of transformation which enable one to change from one system of coördinates to another.

46. Relation of altitude of pole to latitude of observer. It is easy to see that the altitude of the pole equals the latitude ϕ of the observer. For in Figure 18 let NWS be the plane through the center C of the earth parallel to the horizon of the ob-

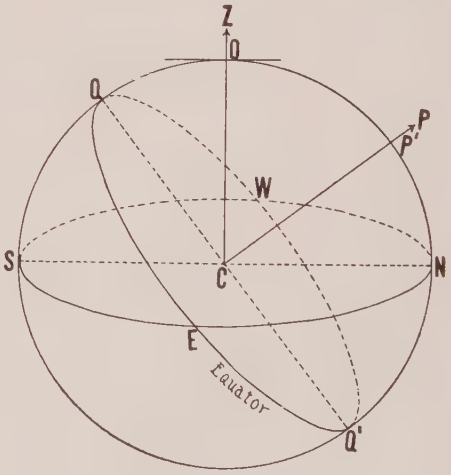

FIG. 18

server at O, $EQ'W$ be the equator, and P' be the terrestrial north pole. Then CO and CP' produced cut the celestial sphere in the zenith Z and the celestial north pole P respectively. Then $ZP = OP' = 90° - QO$ and $OP' = 90° - NP'$. Hence

$$NP' = QO.$$

For an observer south of the equator, of course, the south pole would be employed.

47. The astronomical triangle. The triangle ZPB (zenith-pole-star) in Figure 17 is called the **astronomical triangle.** ZB, the zenith distance of the star, is the complement of the altitude; BP, the polar distance of the star, is the complement of the declination; ZP is the co-latitude of the observer. Angle PZB is numerically equal to the difference between 180° and the azimuth. Angle ZPB is the hour angle of the star. In a triangle each part must be less than 180°. So if t is greater than 180°, angle ZPB must be taken as $360° - t$.

To summarize:

$$\begin{aligned} ZP &= 90° - \phi, \\ ZB &= 90° - h, \\ BP &= 90° - \delta, \\ \angle PZB &= | \, 180° - A \, |, \\ \angle ZPB &= t \text{ or } 360° - t. \end{aligned}$$

If any three of these five coördinates of the star are known, the other two may be found by solving the astronomical triangle.

EXERCISES

(See Appendix E for latitudes and longitudes of places named in the following exercises.)

1. Apparent noon * at any place is that time when the sun crosses the meridian of the place from east to west. The hour angle of the sun at noon is therefore 0°. Show that if t represents the numerical value in degrees

* **Apparent solar time** at any moment is the hour angle of the sun. It is identical with sundial time. Apparent solar days vary in length. Hence ordinary clocks cannot keep sundial time. **Mean solar time** is a compromise to meet this difficulty. The equation of time to be found in the *Nautical Almanac* gives the correction to be added algebraically to the apparent solar time to give the mean solar time.

of the hour angle of the sun at any time, the apparent local time of day is $\frac{t}{15^\circ}$ P.M., or $\frac{t}{15^\circ} - 12$ A.M.

2. In the astronomical triangle, let B represent the sun. Show that you can determine the apparent solar time at any place whose latitude is known, when the declination and altitude of the sun at that time and place are known.

3. Show that if t is the hour angle of a heavenly body, p its polar distance, h its altitude, and ϕ the latitude of the observer

$$\sin \tfrac{1}{2} t = \sqrt{\cos \tfrac{1}{2}(\phi + h + p) \sin \tfrac{1}{2}(\phi - h + p) \sec \phi \csc p},$$

where the acute angle is chosen if the body is west of the meridian.

4. Use the formulas of the first and third exercises to find the apparent solar time of day in Chicago if the sun's declination is 10° N. and its altitude is 41°, the observation being made in the afternoon.

5. Find the apparent time of sunrise in San Francisco if the sun's declination is 21° 58′ N.

6. What point in the celestial sphere has both its right ascension and declination 0°?

7. What are the altitude and azimuth of the zenith? of the north pole?

8. What is the latitude of a place if the celestial equator makes an angle of 50° with the horizon?

9. What are the approximate right ascension and declination of the sun on September 23 (autumnal equinox)?

10. Improvise an instrument for measuring angles and by measuring the altitude of Polaris (approximate north pole), determine your approximate latitude.

11. At 6 A.M. apparent time the sun's altitude was 18°, its declination 20° N. Find the latitude of the place.

12. In what latitude does the sun always set at 6 P.M. apparent time?

13. What is the declination of the sun when it sets at 6 P.M. apparent time at the place where you live?

14. In what direction does the sun set in the preceding exercise?

15. The altitude of a star is 60°, its hour angle is 80°, its azimuth is 135°. Required its declination and the latitude of the observer.

16. Find the latitude of the place at which the sun sets at 9 o'clock on the shortest day of the year. (The declination of the sun is 23° 27′ S. on that day.)

17. The sun's altitude is 38°, its declination is 23° N., and its azimuth is 55°. Find its hour angle and the latitude of the observer, the observation being made in the forenoon.

18. What is the altitude of the sun if a man's shadow is $\frac{1}{2}$ his height? What will be the apparent solar time at 10° north latitude, the sun's declination being 11° S., provided the observation is made in the morning?

19. A star is due west of the observer. Its altitude is 60° and its hour angle 80°. Find its azimuth, declination, and the latitude of the observer.

20. What will be the altitude of the sun at 5 P.M. apparent time in New York if its declination is 10° S.?

21. Show that if l_1 and l_2 represent the numerical values of the longitudes of two places, $\frac{|l_1 \pm l_2|}{15}$ is the number of hours difference in the apparent times of the two places, it being later at the more easterly place, the plus or minus sign being chosen according as the two places are on opposite sides or the same side of the prime meridian.

22. When it is apparent noon in London, what is the apparent time in Moscow?

23. When it is 3 P.M. apparent time in New York, what time is it in Cairo?

24. Show that the local apparent times of sunrise and sunset in terms of the latitude of the observer and the declination of the sun is $\cos t = -\tan \delta \tan \phi$.

25. Show that at 6 A.M. if h is the altitude of the sun $\sin h = \sin \phi \sin \delta$. How does h change on a given day as one goes from the equator to the pole? At what t me of the year is its altitude greatest at a given place?

26. Where and when does the sun rise at Boston on the shortest day of the year? (See ex. 16.)

27. Where and when does the sun set at Boston on the shortest day of the year? (See Fig. 15.)

28. Use the results of the last two exercises to determine the length of the shortest day of the year at Boston.

29. Find the latitude of the place where the sun sets at 8 P.M. apparent time on the longest day of the year. (The declination of the sun is 23° 27′ N. then.)

30. Make up and solve exercises similar to exercises 26, 27, and 28 for the longest day of the year at the place where you live.

31. Given the latitude of the observer and the sun's declination, show that the apparent time when the sun bears due east or west is given by $\cos t = \cot \phi \tan \delta$. Discuss this equation for varying values of ϕ; of δ.

32. What is the apparent solar time when the sun bears due west on the longest day of the year at New Orleans?

33. On January 1 the sun's declination is 23° 3′.9. Find the apparent time of sunset on New Year's day at the place where you live.

HISTORICAL SKETCH

TRIGONOMETRY really originated among the Greeks though they drew on Babylonian and Egyptian sources. That it developed to meet the needs of the astronomers accounts for the remarkable fact that spherical trigonometry made its appearance before plane trigonometry. Hipparchus of Nicæa (161–126 B.C.), the greatest astronomer among the ancients, knew how to solve spherical triangles by arithmetical and graphical methods. The oldest textbook known in spherical trigonometry is the *Sphærica* by Menelaus of Alexandria (about 98 A.D.). This has come down to us only through translation into Arabic, Hebrew, and Latin. In it appeared for the first time a real theory of spherical triangles fully detached from astronomy and developed in a proper sequence like Euclid's *Elements*. Definitions and properties of spherical triangles, the theorem concerning the upper and lower bounds of the sum of the angles of a triangle, and theorems about congruence are given.

The celebrated astronomer, Ptolemy (about 125–151 A.D.), collected the work of his predecessors, improved upon it, and made additions of his own in the *Almagest*, a work which is extant and which governed trigonometric knowledge for a thousand years. It is principally concerned with astronomical investigations. But Ptolemy did not present a system of trigonometry, that is, he gave no rules nor formal methods of solution. He merely showed how to solve particular examples including four of the six cases of right triangles, to the solution of which he reduced all astronomical problems.

After Ptolemy trigonometry passed into other than Greek hands. The Hindus, a people naturally drawn to mathematical and astronomical studies by their religion, had come in contact with Greek learning. While it is possible that they invented the beginnings of trigonometry for themselves, it is probable that they built on the foundations laid by the Greeks. Their writings, which served only

astronomical ends, gave formulas and theorems but no proofs. To them we owe the introduction of the sine, cosine, and versed sine, while the other four fundamental trigonometric functions were introduced by the Arabs.

The Arabs, who had rapidly developed from a wandering desert tribe into a highly cultured people, were the heirs to Greek knowledge and made it their own by many translations and commentaries. Among the East Arabs, Albattani (929 A.D.), a famous astronomer of Bagdad, in his work *De scientia stellarum*, gave all the formulas found in the *Almagest*. Here appears for the first time the fundamental law of cosines for sides of oblique spherical triangles and the law of sines for right triangles. The latter was generalized for oblique triangles by Alhogendi about 970 A.D.

The East and West Arabs were under separate governments. The development of science in Spain was quite independent of that in Persia, due in part to the great distance between Bagdad and Cordova, the center of learning of the West Arabs. Geber of Seville (between 1140 and 1150 A.D.) was the first to treat spherical trigonometry in its continuity. To Ptolemy's four fundamental formulas for the solution of right spherical triangles he added a fifth of his own, $\cos B = \cos b \sin A$, which is often called "Geber's Theorem." Unlike most of the Arabs he gave examples illustrating his theorems. Nasir Eddin (1201–1274) made the last Arab contribution to spherical trigonometry. He treated plane and spherical trigonometry as ends in themselves and not merely as an aid to astronomy. His historical and critical method is of the greatest value. The problems of right and oblique spherical triangles were exhaustively treated and brought to complete solution if not always by the shortest method. He introduced the polar triangle to solve the case when three angles are given, a method which was rediscovered by Vieta in 1593. His great work remained unfruitful and has only recently been rescued from oblivion and accorded the recognition which it deserves.

A new era opened in Europe with Johannes Müller, better known as Regiomontanus (1436–1476). The desire for free and independent judgment in religious matters, which culminated in the reformation, was accompanied by scientific investigation in which

Germany led for a time. Regiomontanus accomplished for the western world what Nasir Eddin had done for the eastern world two hundred years before. His *De triangulis omnimodis* gave the first systematic treatment of trigonometry independent of astronomy in the western world and dominated the later middle ages to a remarkable degree. He took the *Almagest* as his starting point and connected it with the Arabic writings, but the results are so saturated with his spirit as to seem wholly new. He died before completing his great work, which was published a generation later. His manuscript fell into the hands of Johannes Werner (1468–1528). Werner realized its great value, and regretting the fact that it was incomplete started to write a spherical trigonometry himself. The manuscript, though unfinished, exerted an influence on the further development of the subject.

If not entirely independent of the work of his predecessors, Copernicus (1473–1543) developed for himself, in the main, the trigonometric foundations for his astronomical theory in his *De revolutionibus orbium coelestium*. He made it possible to avail himself of plane trigonometry by using the plane triangles formed by drawing perpendiculars from the vertices of the spherical triangle to the radii of the sphere. Though the Arabs had applied the ideas of solid angles in their theorems Copernicus had no knowledge of that fact. His friend Rhaeticus (1514–1576) developed these methods further.

To Vieta (1540–1603), a Frenchman, we owe the first systematic treatment in the western world of the method of solution of spherical triangles based on all six of the trigonometric functions. His sixth formula for the solution of right triangles was new as was also his use of reciprocal representation between general formulas for oblique triangles. Romanus of Louvain (1561–1615) reduced the twenty-eight cases of oblique triangles then considered to only six by means of certain projections. His successors worked for the simplification of computations, an effort which in 1614 culminated in Napier's tables of logarithms. From then on it became necessary to get formulas in which sums and differences were changed into products. Napier (1550–1617), a Scottish philosopher, himself discovered two of the analogies which bear his name, the other two having been

added by Briggs, a contemporary of his. He also gave the rules for circular parts for the solution of right triangles.

Symbolic algebra entered slowly into trigonometry. In the eighteenth century, however, it took a more analytic form. Friedrich Oppel's *Analysis triangulorum* (1746) was the first complete work in analytic trigonometry. The texts of the sixteenth and seventeenth centuries show a tiresome prolixity in handling all possible cases of triangles. Rhaeticus gave a hundred and thirty rules for the solution of the right spherical triangle alone. Indeed in his time it was regarded as a creditable performance. Oppel derived all the formulas for solving spherical triangles from three, the law of sines and the two laws of cosines.

Trigonometry first took on its modern form with Euler (1707–1783), making easier and simpler applications possible. He introduced simple stereographic projections which play an important rôle today. In 1753 he first drew attention to the connection between plane and spherical trigonometry. The generalization of spherical triangles to cases where the sides and angles are not limited occupied Gauss in 1809, Möbius in 1843, and O. Stolz, with the help of analytic geometry, in 1871. E. Study first gave full clarity to modern spherical trigonometry in his fundamental work of 1893.

APPENDIX

APPENDIX

(Formulas are numbered as in text.)

A. *Formulas from plane trigonometry for converting sums and differences of certain trigonometric functions into products, and vice versa.*

$$\sin x + \sin y = 2 \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y)$$
$$\sin x - \sin y = 2 \sin \tfrac{1}{2}(x - y) \cos \tfrac{1}{2}(x + y)$$
$$\cos x + \cos y = 2 \cos \tfrac{1}{2}(x - y) \cos \tfrac{1}{2}(x + y)$$
$$\cos x - \cos y = -2 \sin \tfrac{1}{2}(x - y) \sin \tfrac{1}{2}(x + y)$$

B. *Formulas for the solution of the right spherical triangle.*

$$\cos c = \cos a \cos b \tag{1}$$

$$\begin{cases} \sin b = \sin B \sin c & (2) \\ \sin a = \sin A \sin c & (2') \end{cases}$$

$$\begin{cases} \tan a = \cos B \tan c & (3) \\ \tan b = \cos A \tan c & (3') \end{cases}$$

$$\begin{cases} \tan b = \tan B \sin a & (4) \\ \tan a = \tan A \sin b & (4') \end{cases}$$

$$\cos c = \cot A \cot B \tag{5}$$

$$\begin{cases} \cos B = \sin A \cos b & (6) \\ \cos A = \sin B \cos a & (6') \end{cases}$$

C. *Formulas for the solution of the oblique spherical triangle.*

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \tag{1}$$

$$\begin{cases} \cos a = \cos b \cos c + \sin b \sin c \cos A \\ \cos b = \cos c \cos a + \sin c \sin a \cos B \\ \cos c = \cos a \cos b + \sin a \sin b \cos C \end{cases} \tag{2}$$

$$\begin{cases} \cos A = -\cos B \cos C + \sin B \sin C \cos a \\ \cos B = -\cos C \cos A + \sin C \sin A \cos b \\ \cos C = -\cos A \cos B + \sin A \sin B \cos c \end{cases} \tag{2'}$$

$$\begin{cases} \sin a \cos B = \sin c \cos b - \sin b \cos c \cos A \\ \sin b \cos C = \sin a \cos c - \sin c \cos a \cos B \\ \sin c \cos A = \sin b \cos a - \sin a \cos b \cos C \end{cases} \tag{3}$$

$$\begin{cases} \sin A \cos b = \sin C \cos B + \sin B \cos C \cos a \\ \sin B \cos c = \sin A \cos C + \sin C \cos A \cos b \\ \sin C \cos a = \sin B \cos A + \sin A \cos B \cos c \end{cases} \tag{3'}$$

$$\begin{cases} \sin a \cos C = \sin b \cos c - \sin c \cos b \cos A \\ \sin b \cos A = \sin c \cos a - \sin a \cos c \cos B \\ \sin c \cos B = \sin a \cos b - \sin b \cos a \cos C \end{cases} \tag{4}$$

$$\begin{cases} \sin A \cos c = \sin B \cos C + \sin C \cos B \cos a \\ \sin B \cos a = \sin C \cos A + \sin A \cos C \cos b \\ \sin C \cos b = \sin A \cos B + \sin B \cos A \cos c \end{cases} \tag{4'}$$

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}} \tag{5}$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}} \tag{6}$$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}} \tag{7}$$

$$\tan r = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}$$

$$\tan \tfrac{1}{2} A = \frac{\tan r}{\sin (s-a)} \tag{8}$$

$$\sin \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S-A)}{\sin B \sin C}} \tag{6'}$$

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}} \tag{5'}$$

$$\tan \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}} \tag{9}$$

$$\tan R = \sqrt{\frac{-\cos S}{\cos (S-A) \cos (S-B) \cos (S-C)}}$$

$$\tan \tfrac{1}{2} a = \tan R \cos (S-A) \tag{10}$$

Napier's Analogies.

$$\frac{\tan \tfrac{1}{2}(a-b)}{\tan \tfrac{1}{2} c} = \frac{\sin \tfrac{1}{2}(A-B)}{\sin \tfrac{1}{2}(A+B)} \tag{11}$$

$$\frac{\tan \tfrac{1}{2}(A - B)}{\cot \tfrac{1}{2} C} = \frac{\sin \tfrac{1}{2}(a - b)}{\sin \tfrac{1}{2}(a + b)} \qquad (11')$$

$$\frac{\tan \tfrac{1}{2}(a + b)}{\tan \tfrac{1}{2} c} = \frac{\cos \tfrac{1}{2}(A - B)}{\cos \tfrac{1}{2}(A + B)} \qquad (12)$$

$$\frac{\tan \tfrac{1}{2}(A + B)}{\cot \tfrac{1}{2} C} = \frac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}(a + b)} \qquad (12')$$

Delambre's Analogies.

$$\frac{\sin \tfrac{1}{2}(A - B)}{\cos \tfrac{1}{2} C} = \frac{\sin \tfrac{1}{2}(a - b)}{\sin \tfrac{1}{2} c} \qquad (13)$$

$$\frac{\sin \tfrac{1}{2}(A + B)}{\cos \tfrac{1}{2} C} = \frac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2} c} \qquad (14)$$

$$\frac{\cos \tfrac{1}{2}(A - B)}{\sin \tfrac{1}{2} C} = \frac{\sin \tfrac{1}{2}(a + b)}{\sin \tfrac{1}{2} c} \qquad (14')$$

$$\frac{\cos \tfrac{1}{2}(A + B)}{\sin \tfrac{1}{2} C} = \frac{\cos \tfrac{1}{2}(a + b)}{\cos \tfrac{1}{2} c} \qquad (15)$$

D. *Formulas for the area of a spherical triangle.*

$$\text{area} = E \qquad (2)$$

$$\text{area} = \frac{\pi}{180} ER^2 = E'R^2 \qquad (3)$$

$$\tan \tfrac{1}{4} E^\circ = \sqrt{\tan \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b) \tan \tfrac{1}{2}(s - c) \tan \tfrac{1}{2} s} \qquad (4)$$

E. *A table of latitudes and longitudes.*

Boston 42° 22′ N., 71° 4′ W.
Cairo 30° 5′ N., 31° 17′ E.
Chicago 41° 53′ N., 87° 36′ W.
Eisleben 51° 36′ N., 11° 36′ E.
Gibraltar 36° 6 N., 5° 20′ W.
Honolulu 21° 18′ N., 157° 51′ W.
Liverpool 53° 24′ N., 3° 4′ W.
London 51° 31′ N., 0° W.
Moscow 55° 45′ N., 37° 34′ E.
New Orleans 29° 58′ N., 90° 5′ W.
New York 40° 43′ N., 74° W.
Paris 48° 50′ N., 2° 20′ E.
Pernambuco 8° 3′ N., 34° 55′ W.
Point Barrow 71° 24′ N., 156° 31′ W.
Rio de Janeiro 22° 54′ S., 43° 10′ W.
Sandy Hook 40° 28′ N., 74° 1′ W.
San Francisco 37° 48′ N., 122° 24′ W.
Spitzbergen 80° N., 20° 30′ E.
Tokio 35° 41′ N., 139° 46′ E.

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